This peer review evaluates the manuscript "Functorial Topological Data Compression via Stratified Persistent Sheaves and Enriched Interleavings" by Richard Murdoch Montgomery.

Manuscript Overview

The paper proposes a novel framework for Topological Data Analysis (TDA) by replacing classical persistence modules with **stratified persistent sheaves**. The primary goal is to address the lack of spatial localization in traditional persistent homology. The author leverages the **MacPherson-Treumann equivalence** to provide a combinatorial description of these sheaves via the **exit-path category**, allowing for concrete computations using quiver representations.

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Summary of Peer Review

1. Originality and Significance

The work is highly original. While the connection between sheaves and persistence has been explored by Curry (2014) and Kashiwara & Schapira (2018), this manuscript successfully bridges the gap between abstract microlocal sheaf theory and practical TDA. The introduction of the **enriched interleaving distance** is a significant contribution, as it allows for a more granular comparison of data structures by requiring isomorphisms to respect the underlying stratification.

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2. Technical Soundness and Mathematical Rigor

The mathematical foundations appear robust.

- **Definitions**: The formal definition of a Stratified Persistent Sheaf (SPShv) as a functor from $(\mathbb{R}, \leq)$ to the category of constructible sheaves on a fixed ambient space X is well-motivated. This avoids the "variance ambiguities" often encountered when working with varying sublevel sets.

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- **Stability Theorem**: The proof of the stability theorem (Theorem VI.B) is standard but rigorous, establishing that the enriched distance is bounded by the sup-norm of the filtration functions. This provides the necessary theoretical "safety net" for noise-prone data.

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- **Finite Presentation**: The identification of SPShv with finite quiver representations under specific hypotheses (simply connected strata, finite critical values) is a vital bridge for algorithmic implementation.

3. Critical Areas for Improvement

- **Krull-Schmidt vs. Interval Decomposition**: The author correctly notes that SPShv admit a unique Krull-Schmidt decomposition but **not** necessarily an interval (barcode) decomposition. However, the manuscript would benefit from a more detailed discussion on how to interpret these non-interval indecomposables in a data science context.

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- **Computational Complexity**: While the paper establishes the "existence" of a finite representation, it lacks a formal complexity analysis for computing the enriched interleaving distance between two SPShv.
- **Exit-Path Convention**: The author notes in the revision summary that the exit-path direction was corrected to flow from singular (lower-dimensional) strata to regular (higher-dimensional) strata. While this is the standard mathematical convention, the author should ensure this change is reflected consistently in the diagrams (e.g., Figure 3).

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4. Clarity and Presentation

The organization is logical, following the standard progression from preliminaries to structural theorems and finally a worked example . The inclusion of a detailed example involving a stratified point cloud is helpful for illustrating how features "invisible" to ordinary persistence are captured.

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Recommendation

Accept with Minor Revisions.